ALGORITHM 8
EULER SUMMATION
3 (May 1960), 318 P. NAUR

```
procedure euler (fct, sum, eps, tim) ; value eps, tim ;
integer tim ; real procedure fct ; real sum, eps ;
comment euler computes the sum of fct(i) for i from zero up to
infinity by means of a suitably refined euler transformation. The
summation is stopped as soon as tim times in succession the abso-
lute value of the terms of the transformed series are found to be
less than eps. Hence, one should provide a function fct with one
integer argument, an upper bound eps, and an integer tim. The
output is the sum sum. euler is particularly efficient in the case
of a slowly convergent or divergent alternating series ;
begin integer i, k, n, t ; array m[0:15] ; real mn, mp, ds ;
i := n := t := 0 ; m[0] := fct(0) ; sum := m[0]/2 ;
nextterm: i := i+1 ; mn := fct(i) ;
          for k := 0 step 1 until n do
              begin mp := (mn+m[k])/2 ; m[k] := mn ;
                mn := mp end means ;
          if (abs(mn)<abs(m[n]))∧(n<15) then
              begin ds := mn/2 ; n := n+1 ; m[n] :=
                mn end accept
          else ds := mn ;
          sum := sum + ds ;
          if abs(ds)<eps then t := t+1 else t := 0 ;
          if t<tim then go to nextterm
end euler
```

CERTIFICATION OF ALGORITHM 8
EULER SUMMATION [P. Naur et al. *Comm. ACM* 3, May 1960]

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The body of *euler* was tested on the LGP-30 computer using the Dartmouth SCALP translator. No errors were detected.

The program gave excellent results when used to derive the coefficients for the expansion of *ln* $(1+x)$ in shifted Chebyshev polynomials from the first ten terms of the power series. For $n = 0, 1, 2, 3, 4$, the coefficient of x^i in the power series was multiplied by the coefficient of $T_n^*(x)$ in the expression of x^i in terms of the $T_n^*(x)$. The product, for $i = 1, 2, \cdots, 10$ was used as $fct(i)$ in the program. Results for $n = 0$ were as follows:

i	*fct(i)*	*ds*	*sum*
1	+0.50000000	—	—
2	−0.18750000	+0.07812500	+0.3281250
3	+0.10416667	+0.05729166	+0.3854167
4	−0.068359375	−0.005940758	+0.3794759
5	+0.049218750	−0.001928713	+0.3775471
6	−0.037597656	−0.001357019	+0.3761900
7	+0.029924665	+0.0001742393	+0.3763642
8	−0.024547577	+0.0000571311	+0.3764212
9	+0.020607842	+0.0006395427	+0.3764607
10	−0.017619705	−0.0000055069	+0.3764551
True Value[1]			+0.3764528129.....

Errors less than 0.2×10^{-5} were also found for $n = 1, 2, 3, 4, 5, 6, 7, 8$ and 9.

This technique appears to be a useful supplement to direct telescoping (Algorithms 37 and 38) and to the methods recommended by Clenshaw[1], for slowly convergent power series.

[1] Clenshaw, C. W., *Chebyshev Series for Mathematical Functions.* National Physical Laboratory Math Tables, Vol. 5, London, H.M.S.O. (1962).